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THE METHOD OF SINGULAR PERTURBATIONS IN THE PROBLEM OF STREAM FLOW OVER A RIGHT CIRCULAR CONE

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The laminar stream of incompressible fluid flowing over the surface of a semi-infinite right circular cone at moderate Reynolds numbers is investigated. An asymptotic expansion which takes into account the interaction between the boundary layer and the outer flow is derived. The interaction results in the appearance of a transverse velocity component in the outer inviscid region.

The boundary layer theory does not allow for the derivation of an explicit definition of second order effects in streams, viz. that of the boundary layer and of surface geometry [1]. These effects were taken into account in [2, 3] in the case of plane streams.

The limit case is represented by the propagation of a fan-shaped stream over a plane wall; a self-similar solution of this problem appeared in [4]. Comparison of that solution with experimental data in [5] shows that in the outer region of the boundary layer the former exceeds the latter by some 20%.

The effect of boundary layer displacement and of the cone apex angle on the flow and heat exchange for an axisymmetric laminar stream flowing over a right circular cone is established here using the method of joining asymptotic expansions.

1. Statement of the problem and basic equations. We consider the problem of propagation of an axisymmetric laminar stream of viscous incompressible fluid over the heated surface of a semi-infinite right circular cone with apex half-angle α_0. The stream flows from an infinitely narrow ring source at the cone nose into the space filled with fluid of the same properties as those of the stream. The surface temperature T_w is constant and different from the fluid temperature T_∞ away from the cone. We locate the coordinate origin at the stream source, with the x-axis directed along the cone generatrix and the y-axis normal to it. We use dimensionless equations with the characteristic dimension L taken as the unit of length, projections of velocity on the x- and y-axes are normalized with respect to velocity U at the cross section at the longitudinal coordinate L, and the excess pressure and temperature are normalized with respect to ρU^2 (ρ is the density of the stream fluid) and $T_w - T_\infty$, respectively. Equations and boundary conditions for dimensionless velocity projections u and v, pressure p and temperature Θ in the absence of viscous dissipation and inner heat release are of the form [1]

$$uu_x + vu_y = -p_x + \mathrm{R}^{-1}\{2u_{xx} + 2\varkappa^{-1}u_x + \varkappa^{-1}[\varkappa(v_x + u_y)]_y\} \qquad (1.1)$$
$$uv_x + vv_y = -p_y + \mathrm{R}^{-1}\{\varkappa^{-1}[\varkappa(v_x + u_y)]_x + 2\varkappa^{-1}(\varkappa v_y)_y\}$$

$$u\Theta_x + v\Theta_y = \Pr^{-1}\varkappa^{-1}[(\varkappa\Theta_x)_x + (\varkappa\Theta_y)_y]$$
$$(u\varkappa)_x + (v\varkappa)_y = 0, \quad \varkappa = x + y \operatorname{ctg} \alpha_0$$
$$u = v = 0, \ \Theta = 1, \ y = 0, \ x > 0 \qquad (1.2)$$
$$\psi = \int_0^y u\varkappa \, dy < \infty, \quad \Theta = 0, \quad y = 0, \quad x < 0$$
$$u \to 0, v \to 0, p \to 0, \Theta \to 0, r \to \infty, \alpha \neq 0$$
$$r = \sqrt{x^2 + y^2}, \quad \alpha = \operatorname{arctg}(y/x)$$

With $R \to \infty$ problem (1.1), (1.2) in the terminology of [6] is a problem of singular perturbations. We use $\varepsilon = R^{-1/2}$ as the perturbations parameter.

In conformity with the method of joining asymptotic expansions the flow region is divided into two: the inner (boundary layer region) and the outer regions. For investigating the behavior of solutions in the boundary layer we introduce in the nonuniformity region the transverse coordinate of order unity

$$y = Y\varepsilon \qquad (1.3)$$

In the boundary layer the solution is of the form of asymptotic expansions with $R \to \infty$ and fixed x and Y

$$u = u_0(x, Y) + \varepsilon u_1(x, Y) + \ldots \qquad (1.4)$$
$$v = \varepsilon v_0(x, Y) + \varepsilon^2 v_1(x, Y) + \ldots$$
$$p = p_0(x, Y) + \varepsilon p_1(x, Y) + \ldots$$
$$\Theta = \Theta_0(x, Y) \varepsilon\Theta_1(x, Y) + \ldots$$

Outside the boundary layer the solution is of the form

$$u = U_0(x, y) + \varepsilon U_1(x, y) + \ldots, \ v = V_0(x, y) + \varepsilon V_1(x, y) + \ldots \qquad (1.5)$$
$$p = P_0(x, y) + \varepsilon P_1(x, y) + \ldots, \ \Theta \equiv 0$$

The rule of passing to limit determines the applicability region of each of expansions (1.4) and (1.5).

2. **The zero approximation.** On the assumption of a vortex-free outer flow for the stream function Φ_0 of the outer flow zero approximation form (1.5) and (1.1) we have

$$E^2\Phi_0 = 0 \qquad (2.1)$$
$$E^2 = \partial^2/\partial x^2 - \varkappa^{-1}\partial/\partial x + \partial^2/\partial y^2 - \varkappa^{-1} \operatorname{ctg} \alpha_0 \partial/\partial y$$

Since the boundary conditions are zero, we have without loss of generality

$$\Phi_0(x, y) \equiv 0 \qquad (2.2)$$

The joining procedure determines boundary conditions at the outer boundary of the boundary layer

$$\psi_{0Y}(x, \infty) = 0 \qquad (2.3)$$

The substitution of (1.4) into (1.1) yields a system of equations that defines the boundary layer zero approximation

$$\psi_{0Y}\psi_{0xY} - \psi_{0x}\psi_{0YY} - x^{-1}\psi_{0Y}^2 = x\psi_{0YYY} \qquad (2.4)$$
$$\psi_{0Y}\Theta_{0x} - \psi_{0x}\Theta_{0Y} = \mathrm{Pr}^{-1}x\Theta_{0YY}$$
$$u_0 = x^{-1}\psi_{0Y}, \quad v_0 = -x^{-1}\psi_{0x}$$

which admits the self-similar solution [1] of the form

$$\psi_0(x, Y) = Ax^{1/4}F(\eta), \qquad \Theta_0(x, Y) = H(\eta) \qquad (2.5)$$
$$\eta = BYx^{-5/4}, \quad A = 3^{-1/4}, \ B = 3^{3/4}$$

For the determination of $F(\eta)$ and $H(\eta)$ we have

$$4F''' + FF'' + 2F'^2 = 0, \quad 4H'' + \mathrm{Pr}\,FH = 0 \qquad (2.6)$$

$$F(0) = F'(0) = F'(\infty) = H(\infty) = 0, \quad H(0) = 1, \ \int_0^\infty F'^2F\,d\eta = 1$$

At the outer boundary of the boundary layer

$$F(\eta) \sim F(\infty) + \exp, \quad \eta \to \infty \qquad (2.7)$$

where exp denotes terms that are exponentially small when $\eta \to \infty$.

Using the principle of minimal singularity [8] and the integral invariant E_0 [4] we can express the velocity scale and the Reynolds number in terms of the stream initial characteristics as

$$U = (E_0 / \nu L^3)^{1/4}, \quad R = (E_0 / \nu^3 L)^{1/2} \qquad (2.8)$$

3. The first approximation. For the first approximation of the outer flow stream function Φ_1 we have

$$E^2\Phi_1 = 0 \qquad (3.1)$$

Using (2.7) we determine boundary conditions by the joining procedure

$$\Phi_1(x, 0) = AF(\infty)\,x^{1/4}, \ x > 0; \ \Phi_1(x, 0) < \infty, \ x < 0 \qquad (3.2)$$
$$\Phi_{1x} \to 0, \ \Phi_{1y} \to 0, \ r \to \infty, \ \alpha \neq 0$$

The solution of problem (3.1), (3.2) is of the form

$$\Phi_1(r, \alpha) = AF(\infty)\,r^{1/4}\,\frac{\sin(\alpha + \alpha_0)}{\sin\alpha_0}\,\frac{P_\tau^1[-\cos(\alpha + \alpha_0)]}{P_\tau^1(-\cos\alpha_0)} \qquad (3.3)$$

$$\tau = -\frac{1}{2} + i\frac{\sqrt{11}}{4}$$

where $P_\tau^1(-\cos\alpha)$ are generalized spherical functions of the second kind [7].

The outer boundary conditions for the first approximation of the boundary layer is determined using the joining procedure. Equations and boundary conditions for the inner region are of the form

$$\psi_{1YYY} + x^{-1}\psi_{0x}\psi_{1YY} - x^{-1}\psi_{0Y}\psi_{1xY} - x^{-1}\psi_{0xY}\psi_{1Y} + \qquad (3.4)$$
$$2x^{-2}\psi_{0Y}\psi_{1Y} + x^{-1}\psi_{0YY}\psi_{1x} = \mathrm{ctg}\,\alpha_0\,(Yx^{-3}\psi_{0Y}^2 - Yx^{-1}\psi_{0YYY} +$$
$$x^{-2}\psi_{0x}\psi_{0Y} + x^{-1}\psi_{0YY})$$

$$\psi_{0Y}\Theta_{1x} - \psi_{0x}\Theta_{1Y} - x\mathrm{Pr}^{-1}\Theta_{1YY} = \psi_{1x}\Theta_{0Y} - \psi_{1Y}\Theta_{0x} + x\,\mathrm{ctg}\,\alpha_0\,\mathrm{Pr}^{-1}(Yx^{-1}\Theta_{0YY} + x^{-1}\Theta_{0Y})$$

$$\psi_1 = \psi_{1Y} = \Theta_1 = 0, \quad Y = 0$$

$$\psi_{1Y} \to -AF(\infty)\,x^{-1/4}\frac{P_\tau^2(-\cos\alpha_0)}{P_\tau^1(-\cos\alpha_0)}, \quad \Theta_1 \to 0, \quad Y \to \infty$$

Problem (3.4) admits the self-similar solution

$$\psi_1(x, Y) = AF(\infty)\,x\,\frac{P_\tau^2(-\cos\alpha_0)}{P_\tau^1(-\cos\alpha_0)}\,f(\eta) + x\,\mathrm{ctg}\,\alpha_0 G(\eta) \tag{3.5}$$

$$\Theta_1(x, Y) = AF(\infty)\,x^{1/4}\,\frac{P_\tau^2(-\cos\alpha_0)}{P_\tau^1(-\cos\alpha_0)}\,h(\eta) + x^{1/4}\,\mathrm{ctg}\,\alpha_0 g(\eta)$$

System (3.4) reduces to a system of ordinary differential equations which is then used for determining functions f, G, h, and g. Functions $f(\eta)$ and $h(\eta)$ represent the effect of boundary layer displacement, while $G(\eta)$ and $g(\eta)$ define the effect of the cone surface geometry.

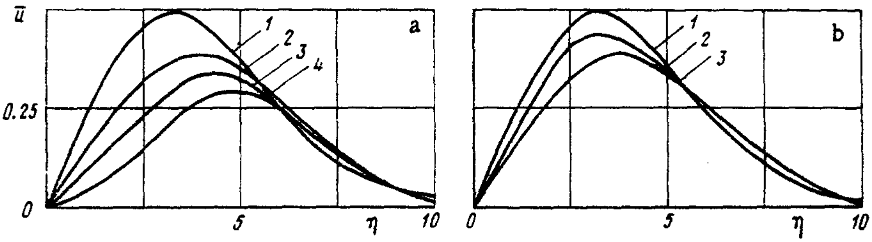

Fig. 1

Owing to the homogeneity of equations and boundary conditions the zero and first approximations for excess pressure in expansions (1.4) and (1.5) are identically zero.

4. **Discussion and comparison with experimental data.** Formulas for the longitudinal and transverse velocity and temperature components are of the form

$$\frac{u^*}{AB\,(E_0/\nu x^*)^{1/2}} = F' + \xi\left[\frac{P_\tau^2(-\cos\alpha_0)}{P_\tau^1(-\cos\alpha_0)}f' + \mathrm{ctg}\,\alpha_0\left(\frac{1}{A}G' - \frac{1}{B}\eta F'\right)\right] \tag{4.1}$$

$$\frac{v^*}{{}^1/_4 A\,(E_0\nu/x^{*5})^{1/4}} = -(3F - 5\eta F') - \xi\left\{\mathrm{ctg}\,\alpha_0\left[3\eta F - 5\eta^2 F' + \frac{A}{4}(G - \eta G')\right] + 4\,\frac{P_\tau^2(-\cos\alpha_0)}{P_\tau^1(-\cos\alpha_0)}(f - \eta f')\right\}$$

$$\Theta = \mathrm{H} + \xi\left[\mathrm{ctg}\,\alpha_0 g + AF(\infty)\,\frac{P_\tau^2(-\cos\alpha_0)}{P_\tau^1(-\cos\alpha_0)}h\right], \quad \xi = \left(\frac{\nu^3 x^*}{E_0}\right)^{1/4}$$

The transverse velocity component induced in the outer flow by the boundary layer displacement action is defined by

$$\frac{v^*}{{}^1/_4 A\,(E_0\nu/x^{*5})^{1/4}} = -\frac{4F(\infty)}{\sin\alpha_0 + B^{-1}\xi\eta\cos\alpha_0}\,(1 + \tag{4.2}$$

$$B^{-2}\xi^2\eta^2)^{-3/2}\,\frac{\sin(\alpha+\alpha_0)}{\sin\alpha_0}\left\{\frac{3}{4}\,\frac{P_\tau^{\,1}[-\cos(\alpha+\alpha_0)]}{P_\tau^{\,1}(-\cos\alpha_0)}-\right.$$
$$\left.B^{-1}\xi\eta\,\frac{P_\tau^{\,2}[-\cos(\alpha-\alpha_0)]}{P_\tau^{\,1}(-\cos\alpha_0)}\right\}$$

Numerical computations of the derived system of ordinary differential equations were carried out using the method of reducing the boundary value problem to that of Cauchy [8], and tables [9] were used for determining the relations of spherical functions. Profiles of dimensionless longitudinal velocities $\bar{u} = u^*[3E_0 / \nu x^{*3})]^{-1/2}$ are shown in Fig. 1, a for several apex half-angles of the cone and fixed parameter ξ. It is seen that with increasing angle α_0 curve gradients increase, and the maximum velocity increases and shifts toward the wall. Curve 1 conforms to the boundary layer theory; curves 2, 3, and 4 correspond to α_0 equal $\pi / 2$, $\pi / 3$, and $\pi / 4$, respectively.

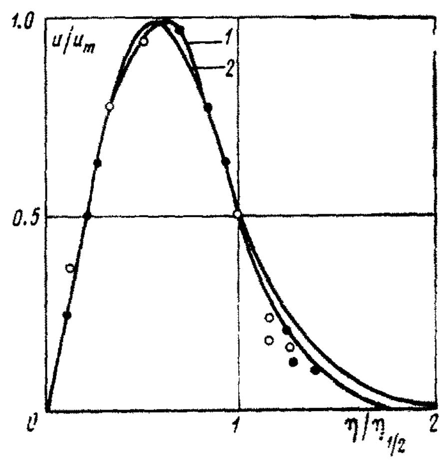

Fig. 2

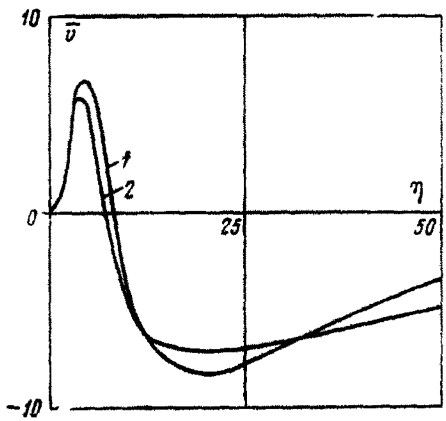

Fig. 3

Profiles of dimensionless longitudinal velocities are shown in Fig. 1, b for several values of parameter ξ in the case of a fan-shaped stream ($\alpha_0 = \pi / 2$). The reduction of the maximum velocity and its recession from the wall with increasing parameter ξ (i.e. with increasing distance from the cone nose with fixed other parameters) can be observed. Curve 1 conforms to the self-similar solution while curves 2 and 3 are for $\xi = 0.005$ and 0.01, respectively.

The universal velocity profile calculated by the described here theory and by that of the boundary layer are shown in Fig. 2 by curves 1 and 2, respectively. The velocity scale is based on the maximum velocity in a given cross section, while that of the transverse coordinate is based on the ordinate at which velocity is equal to half of the maximal velocity. Experimental data obtained with a spreading fan-shaped laminar stream [5] are also shown there (small circles for $R_{x*} = 0.58 \cdot 10^3$, and dots for $R_{x*} = 6.3 \cdot 10^3$ and $R_{x*} = U_s x^* / \nu$). A closer agreement between the proposed theory and experimental data is for $\eta / \eta_{1/2} \geqslant 1.1$.

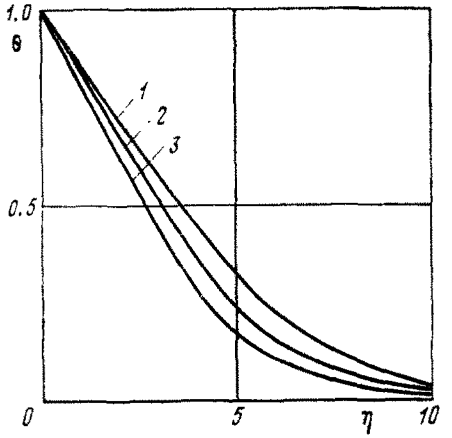

Fig. 4

Profiles of dimensionless transverse velocity components $\bar{v} = 4v^*[(E_0\nu / (3 x^5)]^{-1/4}$ appear in Fig. 3 as functions of parameter ξ (curves 1 and 2 correspond to ξ equal 0.01 and 0.005, respectively) for $\alpha_0 = \pi / 2$. With increasing ξ absolute values of velocity in the inner region increase, while in the outer region they decrease owing to interaction with the wall and surrounding fluid.

Profiles of dimensionless temperature at Prandtl number $\Pr = 0.7$ and several apex half-angles of the cone with fixed ξ are plotted in Fig. 4. Heat transfer from the cone surface becomes more intensive with increasing angle α_0 (curves 1 and 2 correspond to α_0 equal $\pi / 4$ and $\pi / 2$, respectively; curve 3 conforms to the boundary layer theory).

The analysis of longitudinal stream flow over a circular cone shows that displacement of the boundary layer contributes to the decrease of friction stress at the wall and of heat transfer from the cone surface, while the effect of the cone apex half-angle is opposite. The formula for friction stress at the cone surface, with higher approximations taken into account, is of the form

$$\frac{\tau_w}{AB^2\mu\,[E_0^3 / (\nu^5 x^{*11}]^{1/4}} = 0.221 + \xi\left[0.408 \operatorname{ctg} \alpha_0 - 2.206\,\frac{P_\tau^2(-\cos\alpha_0)}{P_\tau^1(-\cos\alpha_0)}\right]$$

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